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LETTER TO THE EDITOR

Wave propagation in a random stratified medium

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Abstract. The transmission coefficient for (electromagnetic) waves propagating in a stochastically stratified medium with small fluctuations of the refractive index is calculated. This coefficient is expressed in terms of the power spectrum of the refractive index. It decreases exponentially with the square root of the layer thickness. As an example the Gaussian correlation function for the refractive index is considered, in which case the transmitted intensity is minimal if the mean wavelength is comparable to the correlation length.

Transmission and/or reflection of (electromagnetic) waves in an inhomogeneous medium depend crucially upon the ratio between the (mean) wavelength and the scale length of the inhomogeneities. Waves with short wavelength see a smoothly varying medium, while long waves essentially do not feel the inhomogeneities. Hence, the propagation of waves will be appreciably influenced by the inhomogeneities of the medium if the ratio of the two quantities is of order unity. In a randomly stratified medium the scale length of the inhomogeneities is given by the correlation length of the refractive index. In the following we shall study the transmission of a monochromatic wave with normal incidence, i.e. we discuss basically the one-dimensional wave propagation in a non-absorbing stochastic medium, whose statistics are assumed to be known.

Since the wavenumber $k = \omega n/c$ is proportional to the refractive index n, the statistics of $k(x)$ are the same as the statistics of $n(x)$. We shall consider small fluctuations only, so that

$$k(x) = k_0(1 + \epsilon f(x)), \tag{1}$$

where $f(x)$ is a given stationary stochastic process with vanishing mean, i.e. $\langle f(x)\rangle = 0$, and with its mean variance equal to unity, i.e. $\langle f^2(x)\rangle = 1$, while k_0 and ϵ are constants.

In principle we have to solve the stochastic wave equation

$$u'' + k^2(x)u = 0, \tag{2}$$

or we must derive an equation for the moments instead (Uscinski 1977). A third possibility is offered if the evolution operator for the field $u(x)$ is known in its explicit dependence upon the function $k(x)$. In this case the statistics of $u(x)$ can be more easily derived from the statistics of k by appropriate averaging of the evolution operator.

No such exact result is known, but we may use an analytic approximation (see Robnik (1979)). In that paper, the variation of the wave amplitudes was described by a transmission matrix whose parameters are expressed explicitly in terms of the function

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$k(x)$. Here we are going to calculate the transmission coefficient

$$|d|^2 = 1/\cosh^2 \tau \tag{3}$$

and the reflection coefficient $|r|^2 = 1 - |d|^2 = \tanh^2 \tau$, so that only the parameter τ is required,

$$\tau = \left| \int_0^L \frac{k'}{2k} \exp\left(2\mathrm{i} \int_0^x k \, \mathrm{d}y\right) \mathrm{d}x \right|. \tag{4}$$

To first order in ϵ, with k from equation (1),

$$\tau = \frac{\epsilon}{2} \int_0^L f'(x) \exp(2\mathrm{i}k_0 x) \, \mathrm{d}x, \tag{5}$$

where $f'(x) := \mathrm{d}f/\mathrm{d}x$. Now for a sufficiently large layer thickness L, the integral above approaches the Fourier transform of the stationary process $\mathrm{d}f/\mathrm{d}x$. If L is larger than the correlation length we may therefore approximate τ^2 by

$$\tau^2 \cong (\epsilon^2/4) L S_{f'}(2k_0),$$

where $S_{f'}(2k_0)$ is the power spectrum of the process $\mathrm{d}f/\mathrm{d}x$ (cf Papoulis 1965), and is simply related to the power spectrum of the given process f, namely by $S_{f'}(2k_0) = 4k_0^2 S_f(2k_0)$. We thereby obtain the final result

$$\tau = \epsilon k_0 (L S_f(2k_0))^{1/2}, \tag{6}$$

which determines the transmission coefficient (3). At large L the intensity of the transmitted waves decays as $|d|^2 \cong 4 \exp[-2(L/s)^{1/2}]$, with the decay length

$$s = 1/(\epsilon^2 k_0^2 S_f(2k_0)). \tag{7}$$

The reason for a non-exponential decreasing of the transmitted intensity is the multiple scattering, i.e. multiple inner reflections. If we neglect them (i.e. if only single scatterings are admitted), we may calculate the transmission coefficient by multiplying the coefficients for infinitesimal layers, which would yield an exponential law in agreement with the result of Uscinski (1977).

As an example we assume a Gaussian correlation function for the medium fluctuations,

$$R(z) := \langle f(x) f(x+z) \rangle = \exp(-z^2/a^2),$$

where a is the correlation length. Then the power spectrum of the process f equals $S_f(2k_0) = a\sqrt{\pi} \exp[-k_0 a)^2]$, and the transmission coefficient for a layer of thickness L reads

$$|d|^2 = 1/\cosh^2\{\pi^{1/4} \epsilon (L/a)^{1/2} (k_0 a) \exp[-(k_0 a)^2/2]\}. \tag{8}$$

We have clearly obtained the expected result: if either $k_0 a \to 0$ or $k_0 a \to \infty$ the transmission approaches unity. Strong reflection of waves due to stochastic inhomogeneities of the medium takes place only if $k_0 a \sim 1$ i.e. if the wavelength is comparable to the correlation length.

It is obvious that for $L \gg s$ (see (7)) the waves can penetrate the medium only by diffusion. To clarify this point let us imagine a monochromatic source placed somewhere in an infinite slab whose refractive index obeys the above statistics. At the distance $L_{1/2} := 0{\cdot}7768\, s$ the transmission probability will be $\frac{1}{2}$, equal to the reflection probability. Hence a random walk (of photons) takes place with step length $L_{1/2}$. The distance x penetrated by the photons after N steps will obey the equation $\langle x^2\rangle = NL_{1/2}^2$. Now, the duration of a single step is $n_0L_{1/2}/c$, so that N equals $ct/n_0L_{1/2}$, where t is time and c/n_0 is the mean velocity of light. The diffusion constant $D := \langle x^2\rangle/t$ is therefore

$$D = cL_{1/2}/n_0 = (0{\cdot}78ac/n_0)/\{\pi^{1/2}\epsilon^2(k_0a)^2 \exp[-(k_0a)^2]\}. \tag{9}$$

The diffusivity assumes the smallest value when the wavelength of the photons is comparable with the correlation length, and is proportional to a/ϵ^2: as the fluctuations of the medium increase, the diffusivity of the waves decreases. Although this analysis of diffusion is certainly oversimplified, one may nevertheless expect it to give a correct qualitative picture.

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